

TOPOLOGICAL SYMMETRY GROUPS FOR SMALL COMPLETE GRAPHS

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ABSTRACT. For each $n \leq 6$, we characterize all the groups which can occur as either the orientation preserving topological symmetry group or the topological symmetry group of some embedding of K_n in S^3 .

Introduction

Chemists have defined the point group of a molecule as its group of rigid symmetries. Though this is useful for rigid molecules, there are molecules with pieces that can twist or rotate independently of the rest of the molecule. In order to classify such symmetries, we use topological symmetry groups.

Definition 1. Let Γ be a graph embedded in S^3 . The **topological symmetry group** of Γ is the subgroup of the automorphism group $\text{Aut}(\Gamma)$ induced by homeomorphisms of the pair (S^3, Γ) . The **orientation preserving topological symmetry group**, $\text{TSG}_+(\Gamma)$, is the subgroup of $\text{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ) .

Though the homeomorphisms in the above definition do not necessarily have finite order, it was shown in [9] that the set of orientation preserving topological symmetry groups of graphs embedded in S^3 is precisely the same as the set of finite subgroups of $\text{SO}(4)$ up to isomorphism. Furthermore it was shown in [8] that the set of orientation preserving topological symmetry groups of complete graphs embedded in S^3 is the same up to isomorphism as the set of finite subgroups of $\text{SO}(3)$ together with the subgroups of products of dihedral groups $D_m \times D_m$ for some odd m . However, this left open the question: for a given n , what topological symmetry groups are possible for embeddings of K_n in S^3 . For $n > 6$, this question was answered for orientation preserving topological symmetry groups in [2], [6], [5], and [7]. In this paper we answer this question for the special cases where $n \leq 6$ and determine both the possible topological symmetry groups and the possible orientation preserving topological symmetry groups of embeddings of K_n in S^3 . We will use the following terminology.

Definition 2. An automorphism f of an abstract graph, γ , is said to be **realizable** if there exists an embedding Γ of γ in S^3 such that f is induced by a homeomorphism of (S^3, Γ) . A group G is said to be **realizable** for γ if there exists an embedding Γ of γ in S^3 such that $\text{TSG}(\Gamma) \cong G$. If $\text{TSG}_+(\Gamma) \cong G$, then we say G is **positively realizable**.

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Not every group is realizable. In particular in [9] it was shown that the alternating group A_q is realizable for some graph if and only if $q \leq 5$.

For each $n \leq 6$, we first determine the orientation preserving topological symmetry groups of embeddings of K_n and then use the fact that either $\text{TSG}_+(\Gamma) = \text{TSG}(\Gamma)$ or $\text{TSG}_+(\Gamma)$ is a normal subgroup of $\text{TSG}(\Gamma)$ of index 2 to help determine the topological symmetry groups. The graph of K_1 is a vertex and of K_2 is an edge. If Γ is an embedding of K_1 , then $\text{TSG}_+(\Gamma)$ and $\text{TSG}(\Gamma)$ are trivial, and if Γ is an embedding of K_2 , then $\text{TSG}_+(\Gamma) = \text{TSG}(\Gamma) \cong \mathbb{Z}_2$. Thus we only consider $n = 3, 4, 5$ and 6.

Let T denote an unknotted embedding of a triangle in S^3 . Then $\text{TSG}(T) = \text{TSG}_+(T) = \text{Aut}(K_3) \cong D_3$. Recall that the trefoil knot 3_1 is chiral while the knot 8_{17} is negative achiral and non-invertible. Therefore by adding the knot 8_{17} to T we obtain an embedding, Γ , such that no orientation preserving homeomorphism of (S^3, Γ) inverts Γ , but there is an orientation reversing homeomorphism which inverts Γ . If Γ contains $8_{17} \# 3_1$, then there is no homeomorphism of (S^3, Γ) which inverts Γ . Note that no matter how K_3 is embedded as Γ , $\text{TSG}_+(\Gamma)$ will never be the trivial group or \mathbb{Z}_2 , because Γ can always be “slithered” along itself to induce an automorphism of order 3. Table 1 summarizes all of the groups that can occur as $\text{TSG}_+(\Gamma)$ or $\text{TSG}(\Gamma)$ for some embedding Γ of K_3 in S^3 .

Knots in Γ	$\text{TSG}(\Gamma)$	$\text{TSG}_+(\Gamma)$
None	D_3	D_3
8_{17}	D_3	\mathbb{Z}_3
$8_{17} \# 3_1$	\mathbb{Z}_3	\mathbb{Z}_3

TABLE 1. TSG Summary for K_3

We will use the following results in our analysis of the topological symmetry groups of embeddings of K_4 , K_5 , and K_6 .

Complete Graph Theorem. [8] *A finite group H is isomorphic to $\text{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is a finite cyclic group, a dihedral group, A_4 , S_4 , A_5 , or a subgroup of $D_m \times D_m$ for some odd m .*

A₄ Theorem. [5] *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\text{TSG}_+(\Gamma) \cong A_4$ if and only if $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.*

A₅ Theorem. [5] *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\text{TSG}_+(\Gamma) \cong A_5$ if and only if $m \equiv 0, 1, 5, 20 \pmod{60}$.*

S₄ Theorem. [5] *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\text{TSG}_+(\Gamma) \cong S_4$ if and only if $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.*

Subgroup Theorem. [6] *Let Γ be an embedding of a 3-connected graph, γ , in S^3 which has an edge that is not pointwise fixed by any non-trivial element of $\text{TSG}_+(\Gamma)$. Then every subgroup of $\text{TSG}_+(\Gamma)$ is positively realizable for γ .*

By [6], adding a knot to an edge of a 3-connected graph is well-defined. Thus if $n > 3$ then for any embedding of K_n in S^3 adding a distinct knot to each edge will create an embedding Δ where $\text{TSG}(\Delta)$ and $\text{TSG}_+(\Delta)$ are both trivial. Thus we do not include the trivial group in our analysis.

Topological Symmetry Groups of K_4

The following is a complete list of all the non-trivial subgroups of $\text{Aut}(K_4) \cong S_4$:

$$S_4, A_4, D_4, D_3, D_2, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2$$

Consider the embedding Γ of K_4 illustrated in Figure 1. The square $\overline{1234}$ must go to itself under any homeomorphism of (S^3, Γ) . Hence $\text{TSG}_+(\Gamma)$ is a subgroup of D_4 . In order to obtain the automorphism (1234) , we rotate the square $\overline{1234}$ clockwise by 90° and pull $\overline{24}$ under $\overline{13}$. We can obtain the transposition (13) by first rotating the figure by 180° about the axis which contains vertices 2 and 4 and then pulling $\overline{24}$ under $\overline{13}$. Thus $\text{TSG}_+(\Gamma) \cong D_4$. Furthermore, since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\text{TSG}_+(\Gamma)$, by the Subgroup Theorem the groups \mathbb{Z}_4 , D_2 and \mathbb{Z}_2 are each positively realizable for K_4 .

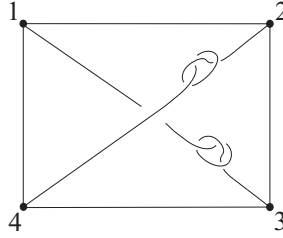


FIGURE 1. $\text{TSG}_+(\Gamma) \cong D_4$.

Next, consider the embedding, Γ of K_4 illustrated in Figure 2. All homeomorphisms of (S^3, Γ) fix vertex 4. The automorphism (123) is realized by a rotation. Also the automorphism (12) is induced by turning the figure upside down and then pushing vertex 4 back up through the centre of $\overline{123}$. Thus $\text{TSG}_+(\Gamma) \cong D_3$. Since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\text{TSG}_+(\Gamma)$, by the Subgroup Theorem \mathbb{Z}_3 is also positively realizable for K_4 .

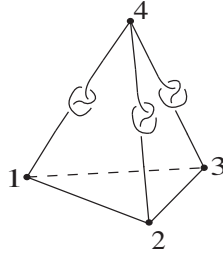


FIGURE 2. $\text{TSG}_+(\Gamma) \cong D_3$.

By adding an identical chiral knot (whose mirror image does not occur in Γ) to every edge of an embedding Γ we get an embedding Γ' such that $\text{TSG}(\Gamma') = \text{TSG}_+(\Gamma)$. Thus every subgroup of S_4 occurs both as $\text{TSG}_+(\Gamma)$ and as $\text{TSG}(\Gamma')$.

Subgroup	Realizable/Positively Realizable	Reason
S_4	Yes	By S_4 Theorem
A_4	Yes	By A_4 Theorem
D_4	Yes	See Figure 1 and argument
D_3	Yes	See Figure 2 and argument
D_2	Yes	By Subgroup Theorem
\mathbb{Z}_4	Yes	By Subgroup Theorem
\mathbb{Z}_3	Yes	By Subgroup Theorem
\mathbb{Z}_2	Yes	By Subgroup Theorem

TABLE 2. Positive realizability of groups for K_4

Topological Symmetry Groups of K_5

The following is a complete list [12] of all the non-trivial subgroups of $\text{Aut}(K_5) \cong S_5$:

$$S_5, A_5, S_4, A_4, \mathbb{Z}_5 \rtimes \mathbb{Z}_4, D_6, D_5, D_4, D_3, D_2, \mathbb{Z}_6, \mathbb{Z}_5, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2$$

Finite Order Theorem. [3] *If φ is an automorphism of K_n with $n > 3$ and Γ is an embedding of K_n in S^3 such that φ is induced by a homeomorphism, h of (S^3, Γ) , then for some re-embedding Γ' of K_n , φ is induced by a finite order homeomorphism, f of (S^3, Γ) , and f is orientation reversing if and only if h is orientation reversing.*

The following lemma follows from the Finite Order Theorem and Smith Theory [14].

Lemma 1. *Let $n > 3$ and let φ be a non-trivial automorphism of K_n which is induced by a homeomorphism h for some embedding of K_n in S^3 . If h is orientation reversing then φ fixes at most 4 vertices. If h is orientation preserving then φ fixes at most 3 vertices. If h is orientation preserving and φ has even order then φ fixes at most 2 vertices.*

Lemma 2. *Let $n > 3$ and let Γ be an embedding of K_n in S^3 such that $\text{TSG}_+(\Gamma)$ contains an element φ of even order $m > 2$. Then φ does not fix any vertex or interchange any pair of vertices.*

Proof. Suppose Γ is an embedding of K_n in S^3 such that $\varphi \in \text{TSG}_+(\Gamma)$. By the Finite Order Theorem, K_n can be re-embedded as Γ' so that φ is induced on Γ' by a finite order orientation preserving homeomorphism h .

Suppose that φ fixes a vertex or interchanges a pair of vertices of Γ' . Then h must fix some point of Γ . By Smith Theory, $\text{fix}(h) \cong S^1$. Let $r = m/2$. Then h^r induces an involution on Γ' which can be written as a product $(a_1 b_1) \dots (a_q b_q)$ of disjoint transpositions of vertices. Now for each i , h^r fixes a point on the edge $\overline{a_i b_i}$. But $\text{fix}(h^r)$ contains

$\text{fix}(h)$ and thus by Smith Theory $\text{fix}(h^r) = \text{fix}(h)$. Hence h fixes a point on each edge $\overline{a_i b_i}$. Thus h induces also $(a_1 b_1) \dots (a_q b_q)$ on Γ' , which contradicts the hypothesis that the order of φ is $m > 2$. \square

By Lemma 2, there is no embedding of K_5 in S^3 such that $\text{TSG}_+(\Gamma)$ contains an element of order 4 or of order 6. Thus $\text{TSG}_+(\Gamma)$ cannot be D_6 , \mathbb{Z}_6 , D_4 or \mathbb{Z}_4 .

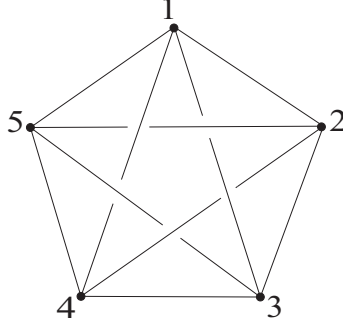


FIGURE 3. $\text{TSG}_+(\Gamma) \cong D_5$.

Consider the embedding Γ of K_5 illustrated in Figure 3. The automorphism (12345) is realizable by rotating Γ , and $(25)(34)$ is induced by turning the graph over. The knotted cycle $\overline{13524}$ must be setwise invariant under every homeomorphism of Γ . Hence $\text{TSG}_+(\Gamma) = \langle (12345), (25)(34) \rangle \cong D_5$. Since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\text{TSG}_+(\Gamma)$, by the Subgroup Theorem the groups \mathbb{Z}_5 and \mathbb{Z}_2 are also positively realizable for K_5 .

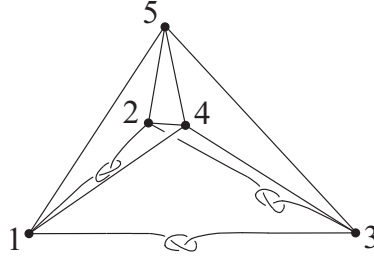


FIGURE 4. $\text{TSG}_+(\Gamma) \cong D_3$.

Next consider the embedding Γ of K_5 illustrated in Figure 4. The triangle $\overline{123}$ must go to itself under any homeomorphism. By Lemma 1, any orientation preserving homeomorphism which fixes vertices 1, 2, and 3 induces a trivial automorphism on K_5 . So $\text{TSG}_+(\Gamma) \leq D_3$. The automorphism (123) is induced by a rotation. Also the automorphism $(45)(12)$ is induced by pulling vertex 4 down through the centre of triangle $\overline{123}$ while pulling vertex 5 into the centre of the figure then rotating by 180° about the line through vertex 3 and the midpoint of the edge $\overline{12}$. Thus $\text{TSG}_+(\Gamma) =$

$\langle (123), (45)(12) \rangle \cong D_3$. Since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\text{TSG}_+(\Gamma)$, by the Subgroup Theorem \mathbb{Z}_3 is positively realizable for K_5 .

Lastly, consider the embedding Γ of K_5 illustrated in Figure 5 with vertex 5 at infinity. The square $\overline{1234}$ must go to itself under any homeomorphism. Hence $\text{TSG}_+(\Gamma) \leq D_4$. The automorphism $(13)(24)$ is induced by rotating the square by 180° . By turning over the figure we can obtain $(12)(34)$. By Lemma 2, $\text{TSG}_+(\Gamma)$ cannot contain an element of order 4. Thus $\text{TSG}_+(\Gamma) = \langle (13)(24), (12)(34) \rangle = D_2$.

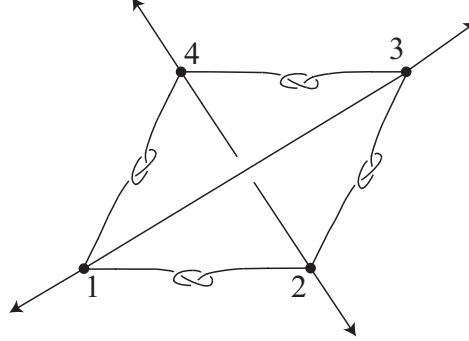


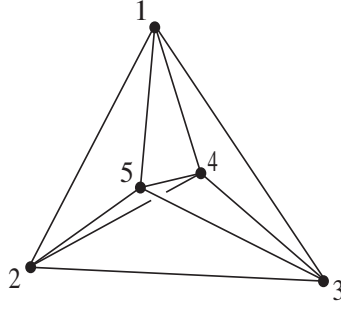
FIGURE 5. $\text{TSG}_+(\Gamma) \cong D_2$.

We summarize our results in Table 3.

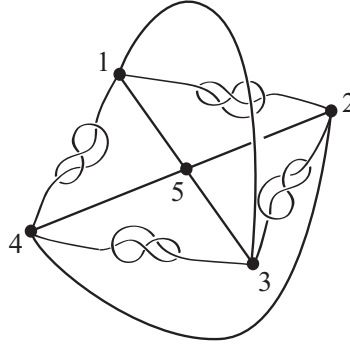
Subgroup	Positively Realizable	Reason
A_5	Yes	By A_5 Theorem
S_4	No	By S_4 Theorem
A_4	Yes	By A_4 Theorem
D_6	No	By Lemma 2
D_5	Yes	See Figure 3 and argument
D_4	No	By Lemma 2
D_3	Yes	See Figure 4 and argument
D_2	Yes	See Figure 5 and argument
\mathbb{Z}_6	No	By Lemma 2
\mathbb{Z}_5	Yes	By Subgroup Theorem
\mathbb{Z}_4	No	By Lemma 2
\mathbb{Z}_3	Yes	By Subgroup Theorem
\mathbb{Z}_2	Yes	By Subgroup Theorem

TABLE 3. Positive realizability of groups for K_5

Again by adding appropriate identical chiral knots to each edge, all of the groups in Table 3 are also realizable. Thus we only need to determine realizability for the groups: S_5 , S_4 , $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, D_6 , D_4 , \mathbb{Z}_6 , and \mathbb{Z}_4 .


 FIGURE 6. $\text{TSG}(\Gamma) \cong S_5$.

Let Γ be the embedding of K_5 illustrated in Figure 6. Any transposition which fixes vertex 5 can be achieved via a reflection in the plane containing the three vertices fixed by the transposition. To see that any transposition involving vertex 5 can be achieved, consider the automorphism (15). Pull $\overline{51}$ through $\overline{234}$ and then turn over the embedding such that vertex 5 is at the top, vertex 1 is in the centre and vertices 3 and 4 are switched. Now reflect in the plane containing vertices 1, 5, and 2 in order to switch back vertices 3 and 4. Similarly, all other transpositions can be achieved. Hence $\text{TSG}(\Gamma) \cong S_5$. Now let Γ' be obtained from Figure 6 by adding the figure eight knot, 4_1 , to all edges containing vertex 5. Every homeomorphism of (S^3, Γ') fixes vertex 5. All transpositions fixing vertex 5 are possible. Thus $\text{TSG}(\Gamma') \cong S_4$.


 FIGURE 7. $\text{TSG}(\Gamma) \cong D_4$.

In order to prove D_4 is realizable for K_5 consider the embedding Γ illustrated in Figure 7. Every homeomorphism of (S^3, Γ) takes $\overline{1234}$ to itself, so $\text{TSG}(\Gamma) \leq D_4$. The automorphism (1234) is induced by rotating the graph by 90° about a vertical line through vertex 5, reflecting in the plane containing the vertices 1, 2, 3, 4, and isotoping the knots into position. Furthermore, reflecting in the plane containing $\overline{153}$ or $\overline{254}$ and then isotoping the knots into position yields the transposition (24) or (13) respectively. Hence $\text{TSG}(\Gamma) \cong D_4$. Now replace the 4_1 knots in Figure 7 with the knot 12_{427} , which is positive achiral and non-invertible [10] to create an embedding, Γ' of K_5 . Since 12_{427} is

not negative achiral or invertible, no homeomorphism of (S^3, Γ) can invert $\overline{1234}$. Thus $\text{TSG}(\Gamma') \cong \mathbb{Z}_4$.

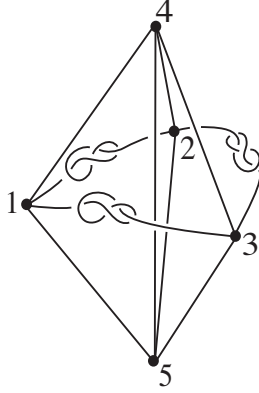


FIGURE 8. $\text{TSG}(\Gamma) \cong D_6$.

Next let Γ denote the embedding of K_5 illustrated in Figure 8. The 3-cycle (123) is induced by a rotation. Transpositions involving only vertices 1, 2 and 3 are all possible via a reflection in the plane containing $\overline{45}$ and the remaining fixed vertex followed by an isotopy. Also the transposition (45) is induced by a reflection in the plane containing vertices 1, 2 and 3 followed by an isotopy. Thus $\text{TSG}(\Gamma) \cong D_6$, generated by (123) , (23) , and (45) . Now if the 4_1 knots in Figure 8 are replaced by 12_{427} then the triangle $\overline{123}$ cannot be inverted. Thus $\text{TSG}(\Gamma) \cong \mathbb{Z}_6$, generated by (123) and (45) .

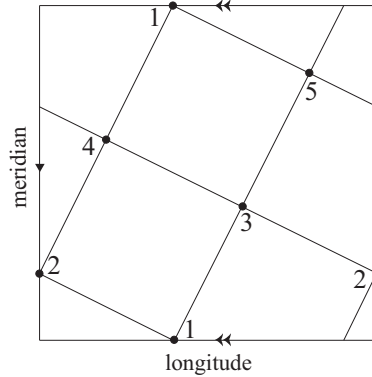


FIGURE 9. Embedding of K_5 in a torus T

In Figure 9 we illustrate an embedding Γ of K_5 on the surface of a torus, T . Let T be standardly embedded in S^3 . Let f denote a glide rotation of S^3 which rotates T by $4\pi/5$ longitudinally and $8\pi/5$ meridinally. Then f takes Γ to itself inducing (12345) . Let g denote the homeomorphism given by rotating S^3 about a $(1, 1)$ curve on T followed by

a reflection through a sphere meeting T in two longitudes, together with a meridional rotation of T by $6\pi/5$. In Figure 10 we illustrate the action of g on T , showing that g takes Γ to itself inducing (2431) .

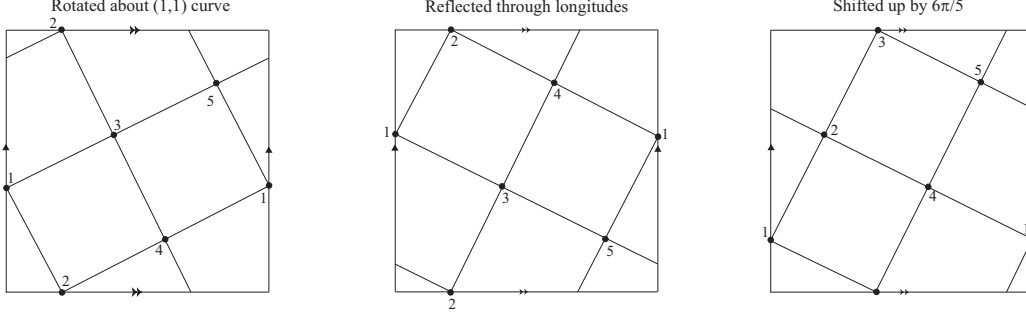


FIGURE 10. The action of g on T and Γ

The homeomorphisms f and g induce the automorphisms $\phi = (12345)$ and $\psi = (2431)$ of Γ . Observe that $\phi^5 = \psi^4 = 1$ and $\psi\phi = \phi\psi^2$. It follows that $\langle \phi, \psi \rangle = \mathbb{Z}_5 \rtimes \mathbb{Z}_4 \leq \text{TSG}(\Gamma)$

In order to obtain the group $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ we modify the embedding in Figure 9 to obtain the embedding Γ' of K_5 illustrated in Figure 11. The pattern in the square $\overline{1534}$ is repeated in each square. Thus f also induces the automorphism $\phi = (12345)$ on Γ' .

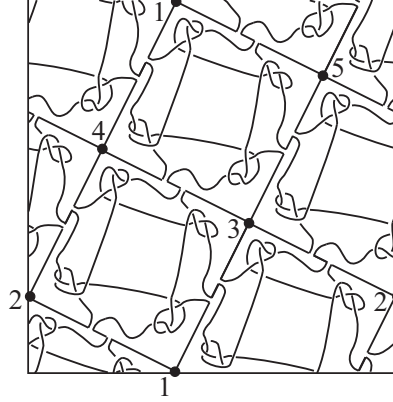
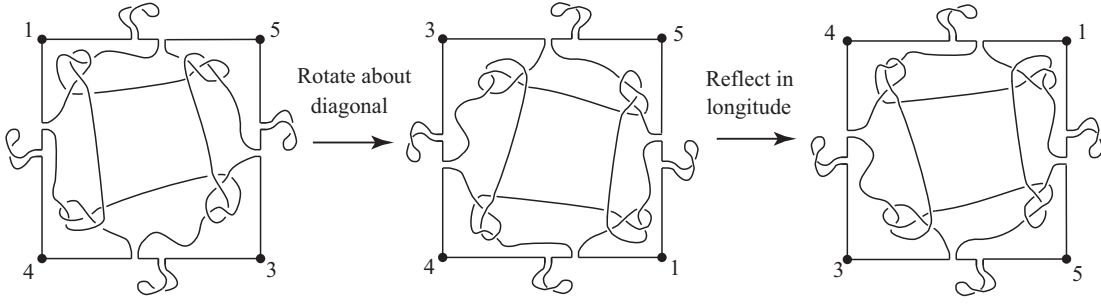


FIGURE 11. Projection Γ' of K_5 on T

Observe from the images in Figure 12 that rotating the square $\overline{1534}$ about a diagonal, then reflecting in a horizontal plane takes the knot $\overline{1534}$ to an equivalent knot. After rotating the torus meridionally by $6\pi/5$, we see that g takes Γ' to itself inducing the automorphism $\psi = (2431)$. Thus $\mathbb{Z}_5 \rtimes \mathbb{Z}_4 \leq \text{TSG}(\Gamma') \leq S_5$.

In Figure 12 we can see that the square $\overline{1534}$ is the connected sum of four figure eight knots. In Figure 13 we highlight the square $\overline{5134}$. In Figure 14 we have isotoped the

FIGURE 12. Effect of g on the square $\overline{1534}$

trivial arcs in order to show that the square $\overline{5134}$ has a projection with 10 crossings, and hence the square $\overline{5134}$ is not the knot $4_1 \# 4_1 \# 4_1 \# 4_1$. It follows that (15) is not induced by a homeomorphism of (S^3, Γ') . However, the only subgroup of S_5 that contains $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ as a proper subgroup is S_5 itself. Thus $\text{TSG}(\Gamma) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

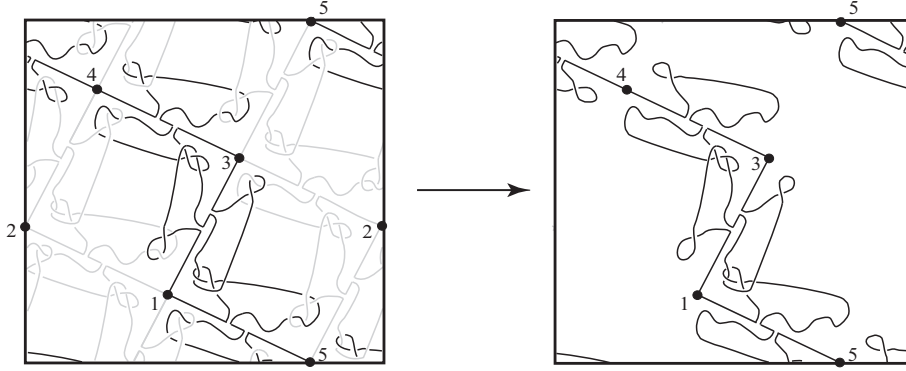
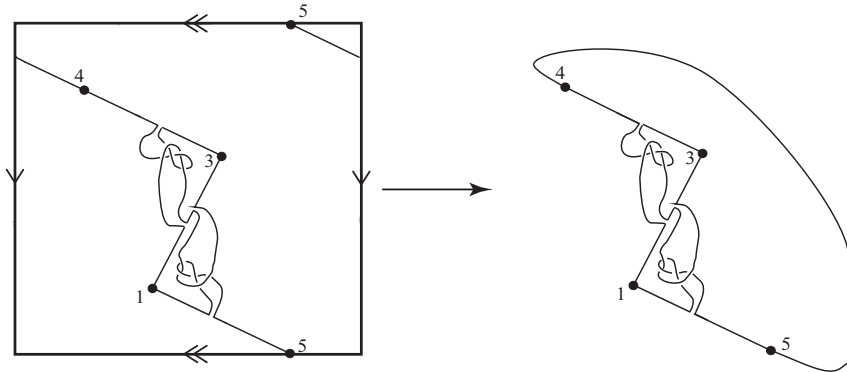
FIGURE 13. Square $\overline{5134}$ highlighted.FIGURE 14. The square $\overline{5134}$ after isotopy removing trivial loops

Table 4 summarizes our results for $\text{TSG}(K_5)$.

Subgroup	Realizable	Reason
S_5	Yes	See Figure 6 and argument
A_5	Yes	By adding chiral knots
S_4	Yes	See Figure 6 and argument
A_4	Yes	By adding chiral knots
D_6	Yes	See Figure 8 and argument
D_5	Yes	By adding chiral knots
D_4	Yes	See Figure 7 and argument
D_3	Yes	By adding chiral knots
D_2	Yes	By adding chiral knots
\mathbb{Z}_6	Yes	See Figure 8 and argument
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	Yes	See Figure 11 and argument
\mathbb{Z}_5	Yes	By adding chiral knots
\mathbb{Z}_4	Yes	See Figure 7 and argument
\mathbb{Z}_3	Yes	By adding chiral knots
\mathbb{Z}_2	Yes	By adding chiral knots

TABLE 4. Realizability for K_5

Topological Symmetry Groups of K_6

The following is a complete list [13] of all the subgroups of $\text{Aut}(K_6) \cong S_6$:

S_6 , A_6 , S_5 , A_5 , $S_3 \wr \mathbb{Z}_2^*$, $S_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, S_4 , A_4 , $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $D_3 \times D_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $D_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, D_6 , D_5 , D_4 , $D_4 \times \mathbb{Z}_2$, D_3 , D_2 , \mathbb{Z}_6 , \mathbb{Z}_5 , \mathbb{Z}_4 , $\mathbb{Z}_4 \times \mathbb{Z}_2$, \mathbb{Z}_3 , \mathbb{Z}_2 , $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

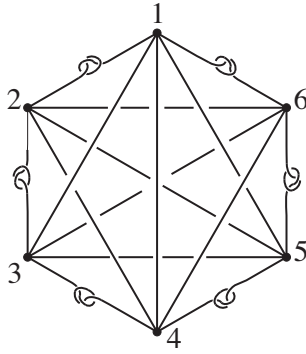


FIGURE 15. $\text{TSG}_+(\Gamma) \cong D_6$.

* $A \wr B$ represents a wreath product of A by B .

Consider the embedding Γ of K_6 illustrated in Figure 15. There are three paths in Γ which look like the letter “Z”, one on top of another. The top Z-path is $\overline{3146}$, the middle Z-path is $\overline{4251}$, and the bottom Z-path is $\overline{5362}$. The knotted cycle $\overline{123456}$ must be setwise invariant under every homeomorphism of Γ . The automorphism (123456) is induced by a glide rotation that cyclically permutes the Z-paths. The automorphism $(13)(46)$ is induced by rotating by 180° about the line through vertices 2 and 5 and then pulling the edges $\overline{13}$ and $\overline{46}$ to the top level while pushing the lower ones down. Thus $\langle (123456), (13)(46) \rangle = D_6 \cong \text{TSG}_+(\Gamma)$. Also since the edge $\overline{12}$ is not pointwise fixed by any non-trivial element of $\text{TSG}_+(\Gamma)$, by the Subgroup Theorem the groups \mathbb{Z}_6 , D_3 , \mathbb{Z}_3 , D_2 and \mathbb{Z}_2 are positively realizable for K_6 .

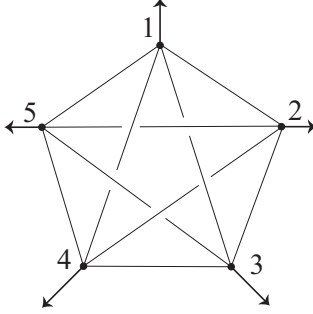


FIGURE 16. $\text{TSG}_+(\Gamma) \cong D_5$.

Consider the embedding, Γ of K_6 illustrated in Figure 16 with vertex 6 at infinity. The automorphisms (13524) and $(25)(34)$ are realizable by rotations. Also since $\overline{13524}$ is the only 5-cycle which is knotted, $\overline{13524}$ is setwise invariant under every homeomorphism of (S^3, Γ) . Hence $\text{TSG}_+(\Gamma) \cong D_5$. Also since $\overline{15}$ is not pointwise fixed under any homeomorphism the Subgroup Theorem, \mathbb{Z}_5 is positively realizable for K_6 .

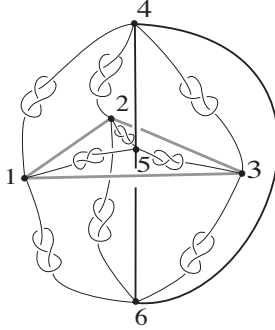


FIGURE 17. $\text{TSG}_+(\Gamma) \cong D_3 \times D_3$.

Next consider the embedding, Γ of K_6 illustrated in Figure 17. The automorphisms $(123)(456)$ and $(123)(465)$ are realizable by glide rotations and $(46)(12)$ is realizable by turning the figure upside down. Also if we consider the circles $\overline{123}$ and $\overline{465}$ as cores of

complementary solid tori, then (14)(25)(36) is realizable by an orientation preserving homeomorphism that switches the two solid tori.

Observe that every homeomorphism of (S^3, Γ) takes the pair of triangles $\overline{123} \cup \overline{456}$ to itself, since this is the only pair of complementary triangles not containing knots. The automorphism group of the union of two triangles is $S_3 \wr \mathbb{Z}_2$ [11]. Note that the automorphisms (12) and (46) are induced by reflections followed by an isotopy. Thus $\text{TSG}(\Gamma) \cong S_3 \wr \mathbb{Z}_2$, since (123)(456), (123)(465), (12) and (14)(25)(36) generate $S_3 \wr \mathbb{Z}_2$. But by the Complete Graph Theorem, $\text{TSG}_+(\Gamma) \neq S_3 \wr \mathbb{Z}_2$ and thus must be an index 2 subgroup of $S_3 \wr \mathbb{Z}_2$ containing $f = (123)(456)$, $g = (123)(465)$, $\phi = (46)(12)$ and $\psi = (14)(25)(36)$. The automorphism f commutes with ψ and anti-commutes with $\phi\psi$, while g commutes with $\phi\psi$ and anti-commutes with ψ . Thus $\text{TSG}_+(\Gamma) \cong D_3 \times D_3$ generated by f, g, ϕ and ψ .

The subgroup $\langle f, g, \psi \rangle \cong D_3 \times \mathbb{Z}_3$ because ψ commutes with f and anti-commutes with g . If we add the non-invertible knot 8_{17} to the edges of the triangles $\overline{123}$ and $\overline{456}$ to obtain an embedding Γ_1 , then the automorphism ϕ is no longer realizable by an orientation preserving homeomorphism. Thus we have an embedding Γ_1 with $\text{TSG}_+(\Gamma_1) \cong D_3 \times \mathbb{Z}_3$.

Also $\langle f, g, \phi \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ because ϕ anti-commutes with both f and g . Again starting with Γ in Figure 17, we place 5_2 knots on the edges of the triangle $\overline{123}$ so that ψ is no longer realizable. Thus creating Γ_2 with $\text{TSG}_+(\Gamma_2) \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Finally $\langle f, g \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If we place identical non-invertible knots on the edges of the triangle $\overline{123}$ and a distinct set of identical non-invertible knots on the edges of $\overline{456}$ we obtain Γ_3 with $\text{TSG}_+(\Gamma_3) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

We summarize our results in Table 5.

Subgroup	Positively Realizable	Reason
A_5	No	By A_5 Theorem
S_4	No	By S_4 Theorem
A_4	No	By A_4 Theorem
D_6	Yes	By Figure 15 and argument
D_5	Yes	By Figure 16 and argument
D_4	No	By Lemma 2
$D_3 \times D_3$	Yes	By Figure 17 and argument
$D_3 \times \mathbb{Z}_3$	Yes	By Subgroup Theorem
D_3	Yes	By Subgroup Theorem
D_2	Yes	By Subgroup Theorem
\mathbb{Z}_6	Yes	By Subgroup Theorem
\mathbb{Z}_5	Yes	By Subgroup Theorem
\mathbb{Z}_4	No	By Lemma 2
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Yes	By Subgroup Theorem
$\mathbb{Z}_3 \times \mathbb{Z}_3$	Yes	By Subgroup Theorem
\mathbb{Z}_3	Yes	By Subgroup Theorem
\mathbb{Z}_2	Yes	By Subgroup Theorem

TABLE 5. Positive realizability of groups for K_6

By adding appropriate identical chiral knots to each edge, every group which is positively realizable for K_6 is also realizable for K_6 . So we evaluate whether or not the groups S_6 , A_6 , S_5 , A_5 , $S_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, S_4 , A_4 , $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, D_4 , $D_4 \times \mathbb{Z}_2$, \mathbb{Z}_4 , $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are realizable. Note that we already showed that $S_3 \wr \mathbb{Z}_2$ is realizable for K_6 .

Let Γ_4 be the embedding of K_6 illustrated in Figure 17 with left handed trefoils added to each edge of $\overline{123}$ and right handed trefoils added to each edge of $\overline{456}$. The pair of triangles are setwise invariant since no other edges contain trefoils. Both $(123)(456)$ and $(123)(465)$ are induced by homeomorphisms of (Γ_4, S^3) . Also if we reflect in the plane containing vertices 1, 4, 5, and 6 then all the trefoils switch from left-handed to right-handed and vice versa. If we then interchange the complementary solid tori with the triangles as cores followed by an isotopy, then we obtain an orientation reversing homeomorphism that induces $(14)(25)(36)(23) = (14)(2536)$. The automorphism $(14)(2536)$ has order 4 and along with $(123)(456)$ and $(123)(465)$ generates $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$.

We see as follows that $\text{TSG}(\Gamma_4)$ cannot be larger than $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$. Suppose (12) is induced by a homeomorphism f . By the Lemma 1, f is orientation reversing. But $f(\overline{456}) = \overline{456}$, which is impossible because $\overline{456}$ contains only right handed trefoils. Thus $(12) \notin \text{TSG}(\Gamma_4)$. Note there is no proper subgroup of $S_3 \wr \mathbb{Z}_2$ containing $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ as a proper subgroup. Thus $\text{TSG}(\Gamma_4) = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$.

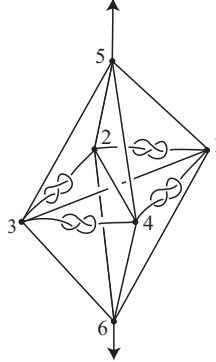


FIGURE 18. $\text{TSG}(\Gamma) \cong D_4$.

Now let Γ be the embedding of K_6 in S^3 illustrated in Figure 18. The automorphism $(1234)(56)$ is induced by a rotation followed by a reflection and an isotopy. In addition the automorphism $(14)(23)(56)$ is induced by turning the figure upside down. Observe that the linking number $\text{lk}(\overline{135}, \overline{246}) = \pm 1$, but $\text{lk}(\overline{136}, \overline{245}) = 0$. Thus (56) is not realizable. Since every homeomorphism of (S^3, Γ) takes $\overline{1234}$ to itself and (56) is not realizable, $\text{TSG}(\Gamma) \leq D_4$. Thus $\text{TSG}(\Gamma) \cong D_4$ generated by $(1234)(56)$ and $(14)(23)(56)$. Now let Γ' be obtained from Γ in Figure 18 by replacing the 4_1 knots with 12_{427} knots. Then the square $\overline{1234}$ can no longer be inverted. In this case $(1234)(56)$ generates $\text{TSG}(\Gamma')$ and thus $\text{TSG}(\Gamma) \cong \mathbb{Z}_4$.

For the next few groups we will use the following lemma.

4-Cycle Theorem. [4] *For any embedding Γ of K_6 in S^3 , and any labelling of the vertices of K_6 by the numbers 1 through 6, there is no homeomorphism of (S^3, Γ) which induces the permutation (1234) on the vertices of K_6 .*

Consider the subgroup $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ of S_6 . Without loss of generality we can assume the order 5 generator is $y = (12345)$. By the 4-Cycle Theorem, for any embedding Γ of K_6 in S^3 , any order 4 element of $\text{TSG}(\Gamma)$ must be of the form $x = (abcd)(ef)$. The presentation of $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ in S_6 gives the relation $x^{-1}yx = y^2$. However, there is no element in S_6 of the form $(abcd)(ef)$ that together with $y = (12345)$ satisfies this relation. Thus there is no embedding Γ of K_6 in S^3 such that $\text{TSG}(\Gamma) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

Now consider the subgroup $\mathbb{Z}_4 \times \mathbb{Z}_2$ of S_6 . By the 4-Cycle Theorem, without loss of generality we may assume that if some embedding Γ of K_6 in S^3 has an element of order 4, then $\text{TSG}(\Gamma)$ contains the element (1234)(56). Computation shows that the only transposition that commutes with (1234)(56) is (56) which cannot be an element of $\text{TSG}(\Gamma)$ since this would imply that (1234) is an element of $\text{TSG}(\Gamma)$. Furthermore the only other order 2 element of S_6 that commutes with (1234)(56) is (13)(24) which is already in the group generated by (1234)(56). Thus there is no embedding Γ of K_6 in S^3 such that $\text{TSG}(\Gamma)$ contains $\mathbb{Z}_4 \times \mathbb{Z}_2$. This rules out the subgroups $S_4 \times \mathbb{Z}_2$, $D_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ as topological symmetry groups.

For the last group we use the following result.

Conway and Gordon Theorem. [1] *For any embedding Γ of K_6 in S^3 , the mod 2 sum of the linking numbers of all pairs of complementary triangles in Γ is 1.*

Suppose that for some embedding Γ of K_6 in S^3 we have $\text{TSG}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Without loss of generality we can assume that $\text{TSG}(\Gamma)$ is generated by (13), (24), and (56) which are induced by homeomorphisms h , f , and g of S^3 respectively. Since any three vertices of Γ determine a pair of disjoint triangles. We write a triple of vertices to represent a pair of disjoint triangles. The orbits of the ten pairs of disjoint triangles in K_6 under the group $\langle (13), (24), (56) \rangle$ are:

$$\langle 123, 143 \rangle, \langle 124, 324 \rangle, \langle 125, 325, 145, 126 \rangle, \langle 135, 136 \rangle$$

Since h , f , and g are homeomorphisms of (S^3, Γ) the links in a given orbit all have the same (mod 2) linking number. Since each orbit has an even number of elements this contradicts the Conway Gordon Theorem. Thus $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is not realizable for K_6 .

Table 6 summarizes our realizability results for K_6 . Observe that for $n = 4$ and $n = 5$ every subgroup of S_n is realizable for K_n , but this is not true for $n = 6$.

Subgroup	Realizable	Reason
S_6	No	$\text{TSG}_+(K_6)$ cannot be S_6 or A_6
A_6	No	$\text{TSG}_+(K_6)$ cannot be A_6
S_5	No	$\text{TSG}_+(K_6)$ cannot be S_5 or A_5
A_5	No	$\text{TSG}_+(K_6)$ cannot be A_5
$S_4 \times \mathbb{Z}_2$	No	$\text{TSG}_+(K_6)$ cannot be $S_4 \times \mathbb{Z}_2$ or S_4
S_4	No	$\text{TSG}_+(K_6)$ cannot be S_4 or A_4
$A_4 \times \mathbb{Z}_2$	No	$\text{TSG}_+(K_6)$ cannot be $A_4 \times \mathbb{Z}_2$ or A_4
A_4	No	$\text{TSG}_+(K_6)$ cannot be A_4
D_6	Yes	By adding chiral knots
D_5	Yes	By adding chiral knots
$D_4 \times \mathbb{Z}_2$	No	$\text{TSG}_+(K_6)$ cannot be $D_4 \times \mathbb{Z}_2$, D_4 , $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
D_4	Yes	See Figure 18 and argument
$S_3 \wr \mathbb{Z}_2$	Yes	See Figure 17 and argument
$D_3 \times D_3$	Yes	By adding chiral knots
$D_3 \times \mathbb{Z}_3$	Yes	By adding chiral knots
D_3	Yes	By adding chiral knots
D_2	Yes	By adding chiral knots
\mathbb{Z}_6	Yes	By adding chiral knots
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	No	Consequence of 4-Cycle Theorem
\mathbb{Z}_5	Yes	By adding chiral knots
$\mathbb{Z}_4 \times \mathbb{Z}_2$	No	Consequence of 4-Cycle Theorem
\mathbb{Z}_4	Yes	See Figure 18 and argument
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$	Yes	See Figure 17 and argument
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Yes	By adding chiral knots
$\mathbb{Z}_3 \times \mathbb{Z}_3$	Yes	By adding chiral knots
\mathbb{Z}_3	Yes	By adding chiral knots
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	No	Consequence of Conway Gordon Theorem
\mathbb{Z}_2	Yes	By adding chiral knots

TABLE 6. Realizability for K_6

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